

Asymptotics for the ratio and the zeros of multiple Charlier polynomials *

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Abstract

We investigate multiple Charlier polynomials and in particular we will use the (nearest neighbor) recurrence relation to find the asymptotic behavior of the ratio of two multiple Charlier polynomials. This result is then used to obtain the asymptotic distribution of the zeros, which is uniform on an interval. We also deal with the case where one of the parameters of the various Poisson distributions depend on the degree of the polynomial, in which case we obtain another asymptotic distribution of the zeros.

1 Introduction

Charlier polynomials $\{C_n^{(a)}, n = 0, 1, 2, \dots\}$ are orthogonal polynomials for the Poisson distribution, i.e.,

$$\sum_{k=0}^{\infty} C_n^{(a)}(k) C_m^{(a)}(k) \frac{a^k}{k!} = 0, \quad n \neq m,$$

where $a > 0$. These polynomials are orthogonal on the positive integers and as a result their zeros are separated by the integers: between two consecutive integers there can be at most one zero of $C_n^{(a)}$. Charlier polynomials have various applications, e.g., in queueing theory [11] and recently [23], in the analysis of the lengths of weakly increasing subsequences of random words [10], and in the totally asymmetric simple exclusion process (TASEP) [2]. Their asymptotic behavior has been studied by Maejima and Van Assche [14], Kuijlaars and Van Assche [12], Rui and Wong [20], Goh [7], Dunster [5] and most recently by Ou and Wong [17] using the Riemann-Hilbert method.

We will investigate multiple Charlier polynomials, which are polynomials of one variable with orthogonality properties with respect to more than one Poisson distribution. Take r Poisson distributions with parameters $a_1, \dots, a_r > 0$ and such that $a_i \neq a_j$ whenever $i \neq j$. Let $\vec{n} = (n_1, n_2, \dots, n_r)$ be a multi-index of size $|\vec{n}| = n_1 + n_2 + \dots + n_r$, then

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the multiple Charlier polynomial $C_{\vec{n}}$ is the monic polynomial of degree $|\vec{n}|$ for which ([1, p. 29–32], [9, p. 632], [21])

$$\sum_{k=0}^{\infty} C_{\vec{n}}(k) k^{\ell} \frac{(a_j)^k}{k!} = 0, \quad \ell = 0, 1, \dots, n_j - 1, \quad j = 1 \dots, r.$$

For $r = 1$ we retrieve the Charlier polynomials. The multiple Charlier polynomials can be obtained using the Rodrigues formula [1, 9, 21]

$$C_{\vec{n}}(x) = (-1)^{|\vec{n}|} \left(\prod_{j=1}^r a_j^{n_j} \right) \Gamma(x+1) \left(\prod_{j=1}^r a_j^{-x} \nabla^{n_j} a_j^x \right) \frac{1}{\Gamma(x+1)} \quad (1.1)$$

where ∇ is the backward difference operator, given by $\nabla f(x) = f(x) - f(x-1)$. An explicit formula for the multiple Charlier polynomials is

$$C_{\vec{n}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-n_1)_{k_1} \cdots (-n_r)_{k_r} (-x)_{k_1+k_2+\dots+k_r} \frac{(-a_1)^{n_1-k_1} (-a_2)^{n_2-k_2} \cdots (-a_r)^{n_r-k_r}}{k_1! k_2! \cdots k_r!}. \quad (1.2)$$

Multiple Charlier polynomials satisfy a number of (higher order) difference equations (Lee [13] and Van Assche [21]). They appear in remainder Padé approximation for the exponential function [19], as common eigenstates of a set of r non-Hermitian oscillator Hamiltonians [15], and we believe that they are related to the orthogonal functions appearing in two speed TASEP (totally asymmetric simple exclusion process) [3].

In this paper we first obtain in Section 2 some properties of the multiple Charlier polynomials, such as the generating function and the nearest neighbor recurrence relations. The zeros of multiple Charlier polynomials are real, positive and separated by the positive integers, as is the case for the usual Charlier polynomials: between two positive integers, there can be at most one zero of a multiple Charlier polynomial (see, e.g., [18, Theorem 3.4]). The largest zero of $C_{\vec{n}}$ is therefore $\geq |\vec{n}| - 1$. In order to prevent the zeros to go to infinity, we will use a scaling and consider the scaled polynomials $P_{\vec{n},N}(x) = C_{\vec{n}}(Nx)/N^{|\vec{n}|}$. One of the main results in this paper is in Section 3 where we obtain the asymptotic behavior of the ratio of two scaled neighboring multiple Charlier polynomials. We use that result in Section 4 to obtain the asymptotic zero distribution of the scaled multiple Charlier polynomials. Another important result is in Section 5 where we give the asymptotic behavior (ratio asymptotics and zero distribution) when one of the parameters depends on the scaling N . This gives a different asymptotic zero distribution which is somewhat more interesting.

2 Some properties of multiple Charlier polynomials

2.1 Generating function

Charlier polynomials have the generating function [4, Ch. VI, Eq. (1.1)]

$$\sum_{n=0}^{\infty} C_n^{(a)}(x) \frac{t^n}{n!} = (1+t)^x e^{-at}, \quad |t| < 1. \quad (2.1)$$

For multiple Charlier polynomials one has a multivariate generating function (with r variables).

Theorem 2.1. *Multiple Charlier polynomials have the following (multivariate) generating function*

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} C_{\vec{n}}(x) \frac{t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r}}{n_1! n_2! \cdots n_r!} = (1 + t_1 + t_2 + \cdots + t_r)^x \exp(-a_1 t_1 - a_2 t_2 - \cdots - a_r t_r). \quad (2.2)$$

Proof. We can use induction on r . For $r = 1$ we have the familiar generating function for Charlier polynomials (2.1).

Suppose the result is true for $r - 1$, then observe that (1.2) implies

$$C_{\vec{n}}(x) = \sum_{k_r=0}^{n_r} C_{\vec{n}-n_r \vec{e}_r}(x - k_r) (-x)_{k_r} (-n_r)_{k_r} \frac{(-a_r)^{n_r-k_r}}{k_r!}.$$

Hence the multivariate generating function is

$$(1 + t_1 + \cdots + t_{r-1})^x \exp(-a_1 t_1 - \cdots - a_{r-1} t_{r-1}) \sum_{n_r=0}^{\infty} \sum_{k_r=0}^{n_r} (1 + t_1 + \cdots + t_{r-1})^{-k_r} (-x)_{k_r} (-n_r)_{k_r} \frac{t_r^{n_r}}{n_r!} \frac{(-a_r)^{n_r-k_r}}{k_r!}.$$

Changing the order of summation gives

$$(1 + t_1 + \cdots + t_{r-1})^x \exp(-a_1 t_1 - \cdots - a_{r-1} t_{r-1}) \sum_{k_r=0}^{\infty} (1 + t_1 + \cdots + t_{r-1})^{-k_r} \binom{x}{k_r} \sum_{n_r=k_r}^{\infty} \frac{t_r^{n_r}}{(n_r - k_r)!} (-a_r)^{n_r-k_r}$$

and by putting $\ell = n_r - k_r$

$$(1 + t_1 + \cdots + t_{r-1})^x \exp(-a_1 t_1 - \cdots - a_{r-1} t_{r-1}) \sum_{k_r=0}^{\infty} (1 + t_1 + \cdots + t_{r-1})^{-k_r} t_r^{k_r} \binom{x}{k_r} \sum_{\ell=0}^{\infty} \frac{t_r^{\ell}}{\ell!} (-a_r)^{\ell}.$$

Now use

$$\sum_{\ell=0}^{\infty} \frac{t_r^{\ell}}{\ell!} (-a_r)^{\ell} = \exp(-a_r t_r)$$

and

$$\sum_{k_r=0}^{\infty} (1 + t_1 + \cdots + t_{r-1})^{-k_r} t_r^{k_r} \binom{x}{k_r} = \left(1 + \frac{t_r}{1 + t_1 + t_2 + \cdots + t_{r-1}}\right)^x$$

to obtain the desired result. \square

The region of convergence of this generating function is a log-convex set in \mathbb{C}^r , which is the case of all power series in several variables, and the series certainly converges whenever $|t_j| < 1/r$ for every $j \in \{1, 2, \dots, r\}$, or when $|t_j| < c_j$ for $1 \leq j \leq r$, where $0 < c_j < 1$ and $\sum_{j=1}^r c_j = 1$. As a corollary, one can obtain an integral representation of the multiple Charlier polynomial, by integrating r times over a closed curve around 0:

$$\frac{C_{\vec{n}}(x)}{n_1! \cdots n_r!} = \frac{1}{(2\pi i)^r} \oint \cdots \oint \frac{(1 + z_1 + \cdots + z_r)^x \exp(-a_1 z_1 - \cdots - a_r z_r)}{z_1^{n_1+1} \cdots z_r^{n_r+1}} dz_1 \cdots dz_r.$$

2.2 Recurrence relations

For multiple orthogonal polynomials there is always a nearest neighbor recurrence relation of the form

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x) \quad (2.3)$$

where $k = 1, \dots, r$ [9, Thm. 23.1.11], [22], and $\vec{e}_k = (0, \dots, 0, 1, 0, \dots, 0)$ is the k th unit vector in \mathbb{N}^r . The recurrence relation for multiple Charlier polynomials was given in [9, p. 632] without proof. Here we will work out the details of the proof.

Theorem 2.2. *The nearest neighbor recurrence relation for multiple Charlier polynomials is*

$$xC_{\vec{n}}(x) = C_{\vec{n}+\vec{e}_k}(x) + (a_k + |\vec{n}|)C_{\vec{n}}(x) + \sum_{j=1}^r n_j a_j C_{\vec{n}-\vec{e}_j}(x). \quad (2.4)$$

Proof. From (1.2) and $(-x)_n = (-1)^n x^n + (-1)^{n-1} \binom{n}{2} x^{n-1} + \cdots$ we find that

$$C_{\vec{n}}(x) = x^{|\vec{n}|} + \delta_{\vec{n}} x^{|\vec{n}|-1} + \cdots,$$

where $\delta_{\vec{n}}$ can be found by taking $(k_1, k_2, \dots, k_r) = (n_1, n_2, \dots, n_r)$, which gives the contribution $-\binom{|\vec{n}|}{2}$ to $\delta_{\vec{n}}$, and for each j with $1 \leq j \leq r$ we get for $(k_1, k_2, \dots, k_r) = (n_1, n_2, \dots, n_j - 1, \dots, n_r)$ the contribution $-a_j n_j$, so that

$$\delta_{\vec{n}} = -\binom{|\vec{n}|}{2} - \sum_{j=1}^r a_j n_j.$$

If we compare the coefficient of $x^{|\vec{n}|}$ in (2.3), then $b_{\vec{n},k} = \delta_{\vec{n}} - \delta_{\vec{n}+\vec{e}_k}$, which for the multiple Charlier polynomials gives $b_{\vec{n},k} = |\vec{n}| + a_k$. For the recurrence coefficients $a_{\vec{n},j}$ we can use [9, Eq. (23.1.23)]

$$a_{\vec{n},j} = \frac{\sum_{k=0}^{\infty} k^{n_j} C_{\vec{n}}(k) a_j^k / k!}{\sum_{k=0}^{\infty} k^{n_j-1} C_{\vec{n}-\vec{e}_j}(k) a_j^k / k!}.$$

The sums can be computed using the Rodrigues formula (1.1): the difference operators $a_i^{-x} \nabla^{n_i} a_i^x$ ($i = 1, 2, \dots, r$) are commuting, so we can first apply $a_j^{-x} \nabla^{n_j} a_j^x$ to find

$$\sum_{k=0}^{\infty} k^{n_j} C_{\vec{n}}(k) \frac{a_j^k}{k!} = (-1)^{|\vec{n}|} \prod_{i=1}^r a_i^{n_i} \sum_{k=0}^{\infty} k^{n_j} (\nabla^{n_j} a_j^k) \left(\prod_{i=1, i \neq j}^r a_i^{-k} \nabla^{n_i} a_i^k \right) \frac{1}{k!}.$$

Now use summation by parts n_k times to find

$$\begin{aligned} \sum_{k=0}^{\infty} k^{n_j} C_{\vec{n}}(k) \frac{a_j^k}{k!} &= (-1)^{|\vec{n}|} \prod_{i=1}^r a_i^{n_i} (-1)^{n_j} \sum_{k=0}^{\infty} (\Delta^{n_j} k^{n_j}) a_j^k \left(\prod_{i=1, i \neq j}^r a_i^{-k} \nabla^{n_i} a_i^k \right) \frac{1}{k!} \\ &= (-1)^{|\vec{n}|} \prod_{i=1}^r a_i^{n_i} (-1)^{n_j} n_j! \sum_{k=0}^{\infty} a_j^k \left(\prod_{i=1, i \neq j}^r a_i^{-k} \nabla^{n_i} a_i^k \right) \frac{1}{k!}. \end{aligned}$$

If we change n_j to $n_j - 1$ then this gives

$$\begin{aligned} \sum_{k=0}^{\infty} k^{n_j-1} C_{\vec{n}-\vec{e}_j}(k) \frac{a_j^k}{k!} &= (-1)^{|\vec{n}|-1} \left(\prod_{i=1}^r a_i^{n_i} \right) a_j^{-1} (-1)^{n_j-1} (n_j - 1)! \sum_{k=0}^{\infty} a_j^k \left(\prod_{i=1, i \neq j}^r a_i^{-k} \nabla^{n_i} a_i^k \right) \frac{1}{k!}. \end{aligned}$$

Dividing both expressions then gives

$$a_{\vec{n},j} = n_j a_j.$$

□

The recurrence coefficients are quite simple in this case, and in particular $a_{\vec{n},j} = n_j a_j > 0$ whenever $n_j \in \mathbb{N}$. This implies that the zeros of $C_{\vec{n}}$ and its nearest neighbors $C_{\vec{n}+\vec{e}_k}$ interlace for every $k \in \{1, 2, \dots, r\}$, see [8]. This will be useful in the next section.

3 Ratio asymptotics

There are various levels of asymptotic behavior to consider. In this paper we limit the analysis to ratio asymptotic behavior, i.e., the asymptotic behavior of the ratio of two neighboring polynomials. In order to prevent the zeros from going to infinity, we use a scaling and we will investigate the ratio $C_{\vec{n}+\vec{e}_k}(Nx)/C_{\vec{n}}(Nx)$ for $x \in \mathbb{C} \setminus [0, \infty)$, where N is of the order $|\vec{n}|$, i.e., $\lim_{N \rightarrow \infty} |\vec{n}|/N = t > 0$.

Theorem 3.1. *Suppose $n_j = \lfloor q_j n \rfloor$, with $0 < q_j < 1$ and $\sum_{j=1}^r q_j = 1$, so that $|\vec{n}|/n \rightarrow 1$ as $n \rightarrow \infty$. Let $a_i > 0$ for $1 \leq i \leq r$ and $a_i \neq a_j$ whenever $i \neq j$. Then for $t > 0$ and for every $k \in \{1, 2, \dots, r\}$ one has*

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \frac{C_{\vec{n}+\vec{e}_k}(Nx)}{NC_{\vec{n}}(Nx)} = x - t \quad (3.1)$$

uniformly for $x \in K$, where K is a compact set in $\mathbb{C} \setminus [0, \infty)$.

Proof. We will use the notation $P_{\vec{n},N}(x) = C_{\vec{n}}(Nx)/N^{|\vec{n}|}$ for the monic and rescaled multiple Charlier polynomials. The zeros of $C_{\vec{n}-\vec{e}_j}$ and $C_{\vec{n}}$ are real, positive and interlace (since $a_{\vec{n},j} = a_j n_j > 0$ whenever $n_j > 0$, see [8]), hence we have the partial fractions decomposition

$$\frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} = \sum_{i=1}^{|\vec{n}|} \frac{A_{\vec{n},i}}{x - x_{\vec{n},i}/N},$$

where $\{x_{\vec{n},i} : 1 \leq i \leq |\vec{n}|\}$ are the zeros of $C_{\vec{n}}$ and $A_{\vec{n},i} > 0$ for every $i \leq |\vec{n}|$. Let K be a compact set in $\mathbb{C} \setminus [0, \infty)$, then for $x \in K$ we have that

$$\left| \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} \right| \leq \sum_{i=1}^{|\vec{n}|} \frac{A_{\vec{n},i}}{|x - x_{\vec{n},i}/N|} \leq \frac{1}{\delta} \sum_{i=1}^{|\vec{n}|} A_{\vec{n},i},$$

where

$$\delta = \inf\{|z - y| : z \in K, y \in [0, \infty)\} > 0$$

is the minimal distance between K and $[0, \infty)$. Since $P_{\vec{n},N}$ and $P_{\vec{n}-\vec{e}_j,N}$ are monic polynomials, one has $\sum_{i=1}^{|\vec{n}|} A_{\vec{n},i} = 1$, so that we have the bound

$$\left| \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} \right| \leq \frac{1}{\delta}, \quad (3.2)$$

uniformly for $x \in K$. Take the recurrence relation (2.4) with x replaced by Nx , and divide by $C_{\vec{n}}(Nx)$, which is allowed since $x \in K$ cannot be a zero, then we find

$$x = \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} + \frac{a_k + |\vec{n}|}{N} + \sum_{j=1}^r \frac{n_j a_j}{N^2} \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)}.$$

If we use the bound (3.2), then this gives

$$\left| \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} - x + \frac{a_k + |\vec{n}|}{N} \right| \leq \frac{1}{\delta} \sum_{j=1}^r \frac{n_j a_j}{N^2}.$$

Clearly, when $n, N \rightarrow \infty$ in such a way that $n/N \rightarrow t$, we have

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \frac{a_k + |\vec{n}|}{N} = \lim_{n \rightarrow \infty, n/N \rightarrow t} \frac{|\vec{n}|}{n} \frac{n}{N} = t,$$

and

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \frac{a_j n_j}{N^2} = a_j \frac{n_j}{n} \frac{n}{N^2} = 0,$$

so that

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} = x - t,$$

uniformly for $x \in K$, which proves the theorem. \square

Observe that the same result will hold for any family of multiple orthogonal polynomials for which $a_{\vec{n},j} > 0$ whenever $n_j > 0$ and

$$\lim_{n \rightarrow \infty} \frac{b_{\vec{n},k}}{n} = 1, \quad \lim_{n \rightarrow \infty} \frac{a_{\vec{n},j}}{n^2} = 0,$$

where $n_j = \lfloor q_j n \rfloor$, with $0 < q_j < 1$ and $\sum_{j=1}^r q_j = 1$. The fact that $a_{\vec{n},j}/n^2 \rightarrow 0$ simplifies the asymptotic analysis a lot and the limit function is an easy polynomial function of degree 1. In general, the asymptotic analysis for ratios of multiple orthogonal polynomials would involve a limit function which is the solution of an algebraic equation of degree $r+1$.

4 Asymptotic distribution of the zeros

Next, we will obtain the asymptotic distribution of the (scaled) zeros of the multiple Charlier polynomials. For this, we introduce the zero measure

$$\nu_{n,N} = \frac{1}{|\vec{n}|} \sum_{i=1}^{|\vec{n}|} \delta_{x_{\vec{n},i}/N}$$

and we want to show that these (probability) measures converge weakly to a (probability) measure ν_t as $n, N \rightarrow \infty$ and $n/N \rightarrow t > 0$, which then describes the asymptotic distribution of the zeros. Again we will take multi-indices \vec{n} such that $n_j = \lfloor nq_j \rfloor$, where $0 < q_j < 1$ and $\sum_{j=1}^r q_j = 1$, so that $|\vec{n}|/n \rightarrow 1$ as n tends to infinity. In order to prove this weak convergence, we will investigate their Stieltjes transform

$$\int \frac{d\nu_{n,N}(y)}{x-y} = \frac{1}{|\vec{n}|} \frac{P'_{\vec{n},N}(x)}{P_{\vec{n},N}(x)}, \quad x \in \mathbb{C} \setminus [0, \infty),$$

where $P_{\vec{n},N}(x) = C_{\vec{n}}(Nx)/N^{|\vec{n}|}$, and show that they converge to a function, which we can identify as the Stieltjes transform of a measure ν_t . The Grommer-Hamburger theorem [6] then tells us that the measures $\nu_{n,N}$ converge weakly to ν_t as $n, N \rightarrow \infty$ and $n/N \rightarrow t$.

Theorem 4.1. *Suppose $n_j = \lfloor q_j n \rfloor$, with $0 < q_j < 1$ and $\sum_{j=1}^r q_j = 1$ and that $a_i > 0$ for $1 \leq i \leq r$ and $a_i \neq a_j$ whenever $i \neq j$. Let $x_{\vec{n},1} < x_{\vec{n},2} < \dots < x_{\vec{n},|\vec{n}|}$ be the zeros of $C_{\vec{n}}$. Then*

$$\lim_{n,N \rightarrow \infty, n/N \rightarrow t} \frac{1}{|\vec{n}|} \sum_{j=1}^{|\vec{n}|} f(x_{\vec{n},j}/N) = \frac{1}{t} \int_0^t f(x) dx \quad (4.1)$$

for every bounded continuous function on $[0, \infty)$. This means that the zeros of $C_{\vec{n}}(Nx)$ are asymptotically uniform on the interval $[0, t]$ when $n, N \rightarrow \infty$ and $n/N \rightarrow t > 0$.

Proof. We will prove that

$$\lim_{n,N \rightarrow \infty, n/N \rightarrow t} \frac{1}{|\vec{n}|} \frac{P'_{\vec{n},N}(x)}{P_{\vec{n},N}(x)} = \frac{1}{t} \int_0^t \frac{1}{x-y} dy \quad (4.2)$$

uniformly for $x \in K$, where K is a compact set in $\mathbb{C} \setminus [0, \infty)$, which by the Grommer-Hamburger theorem (see, e.g., [6]) is equivalent with the weak convergence to the uniform measure on $[0, t]$. We will prove this by induction on r . For $r = 1$ we deal with the zeros of Charlier polynomials and the multi-index \vec{n} is an integer which we denote by n . Observe that

$$\frac{1}{n} \frac{P'_{n,N}(x)}{P_{n,N}(x)} = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{P'_{k+1,N}(x)}{P_{k+1,N}(x)} - \frac{P'_{k,N}(x)}{P_{k,N}(x)} \right)$$

and straightforward calculus gives

$$\frac{P'_{k+1,N}(x)}{P_{k+1,N}(x)} - \frac{P'_{k,N}(x)}{P_{k,N}(x)} = \left(\frac{P_{k+1,N}(x)}{P_{k,N}(x)} \right)' / \left(\frac{P_{k+1,N}(x)}{P_{k,N}(x)} \right).$$

Hence we may write

$$\frac{1}{n} \frac{P'_{n,N}(x)}{P_{n,N}(x)} = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{P_{k+1,N}(x)}{P_{k,N}(x)} \right)' / \left(\frac{P_{k+1,N}(x)}{P_{k,N}(x)} \right).$$

We can rewrite the sum as an integral by putting $k = \lfloor ns \rfloor$, so that

$$\frac{1}{n} \frac{P'_{n,N}(x)}{P_{n,N}(x)} = \int_0^1 \left(\frac{P_{\lfloor ns \rfloor + 1, N}(x)}{P_{\lfloor ns \rfloor, N}(x)} \right)' / \left(\frac{P_{\lfloor ns \rfloor + 1, N}(x)}{P_{\lfloor ns \rfloor, N}(x)} \right) ds.$$

Now we let $n, N \rightarrow \infty$ in such a way that $n/N \rightarrow t$, and we use Theorem 3.1 (with $r = 1$) to find that uniformly for $x \in K$ (K a compact set in $\mathbb{C} \setminus [0, \infty)$)

$$\lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{1}{n} \frac{P'_{n,N}(x)}{P_{n,N}(x)} = \int_0^1 \frac{(x - st)'}{x - st} ds,$$

where the $'$ in the integral is a derivative with respect to the variable x . The integral on the right is (use $st = y$)

$$\int_0^1 \frac{1}{x - st} ds = \frac{1}{t} \int_0^t \frac{1}{x - y} dy,$$

which proves (4.2) for $r = 1$.

Now suppose that (4.2) is true for $r - 1$. Observe that

$$\frac{P'_{\vec{n}, N}(x)}{P_{\vec{n}, N}(x)} = \frac{P'_{\vec{n} - n_r \vec{e}_r, N}(x)}{P_{\vec{n} - n_r \vec{e}_r, N}(x)} + \sum_{k=0}^{n_r-1} \left(\frac{P_{\vec{n} - k \vec{e}_r, N}(x)}{P_{\vec{n} - (k+1) \vec{e}_r, N}(x)} \right)' / \left(\frac{P_{\vec{n} - k \vec{e}_r, N}(x)}{P_{\vec{n} - (k+1) \vec{e}_r, N}(x)} \right). \quad (4.3)$$

The multiple orthogonal polynomial $P_{\vec{n} - n_r \vec{e}_r}$ is in fact a multiple orthogonal polynomial with only $r - 1$ measures $(\mu_1, \dots, \mu_{r-1})$, hence we can use the induction hypothesis to find

$$\lim_{n, N \rightarrow \infty} \frac{1}{|\vec{n}| - n_r} \frac{P'_{\vec{n} - n_r \vec{e}_r, N}(x)}{P_{\vec{n} - n_r \vec{e}_r, N}(x)} = \frac{1}{(1 - q_r)t} \int_0^{(1 - q_r)t} \frac{1}{x - y} dy. \quad (4.4)$$

Note that $(|\vec{n}| - n_r)/n \rightarrow 1 - q_r$, which explains the appearance of $1 - q_r$ in the last formula. We can write the sum as an integral by taking $k = \lfloor n_r s \rfloor$:

$$\begin{aligned} \frac{1}{n_r} \sum_{k=0}^{n_r-1} \left(\frac{P_{\vec{n} - k \vec{e}_r, N}(x)}{P_{\vec{n} - (k+1) \vec{e}_r, N}(x)} \right)' / \left(\frac{P_{\vec{n} - k \vec{e}_r, N}(x)}{P_{\vec{n} - (k+1) \vec{e}_r, N}(x)} \right) \\ = \int_0^1 \left(\frac{P_{\vec{n} - \lfloor n_r s \rfloor \vec{e}_r, N}(x)}{P_{\vec{n} - (\lfloor n_r s \rfloor + 1) \vec{e}_r, N}(x)} \right)' / \left(\frac{P_{\vec{n} - \lfloor n_r s \rfloor \vec{e}_r, N}(x)}{P_{\vec{n} - (\lfloor n_r s \rfloor + 1) \vec{e}_r, N}(x)} \right) ds. \end{aligned}$$

Now use Theorem 3.1 to find

$$\begin{aligned} \lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{1}{n_r} \sum_{k=0}^{n_r-1} \left(\frac{P_{\vec{n} - k \vec{e}_r, N}(x)}{P_{\vec{n} - (k+1) \vec{e}_r, N}(x)} \right)' / \left(\frac{P_{\vec{n} - k \vec{e}_r, N}(x)}{P_{\vec{n} - (k+1) \vec{e}_r, N}(x)} \right) \\ = \int_0^1 \frac{(x - (1 - q_r s)t)'}{x - (1 - q_r s)t} ds = \frac{1}{q_r t} \int_{(1 - q_r)t}^t \frac{1}{x - y} dy \quad (4.5) \end{aligned}$$

where the last equality follows after using the substitution $y = (1 - q_r s)t$. Note that $(|\vec{n}| - \lfloor n_r s \rfloor)/n \rightarrow 1 - q_r s$, which explains the factor $1 - q_r s$ in the asymptotic formula. Now combine (4.4) and (4.5) in (4.3) to find

$$\begin{aligned} \lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{1}{|\vec{n}|} \frac{P'_{\vec{n}, N}(x)}{P_{\vec{n}, N}(x)} &= \frac{1 - q_r}{(1 - q_r)t} \int_0^{(1 - q_r)t} \frac{1}{x - y} dy + \frac{q_r}{q_r t} \int_{(1 - q_r)t}^t \frac{1}{x - y} dy \\ &= \frac{1}{t} \int_0^t \frac{1}{x - y} dy, \end{aligned}$$

which proves (4.2). \square

5 Parameters depending on the degree

We get more interesting asymptotics when some of the parameters depend on N and grow together with the degree $|\vec{n}|$. The case where only one parameter depends on N can be worked out in detail.

Theorem 5.1. *Suppose $n_j = \lfloor q_j n \rfloor$, with $0 < q_j < 1$ and $\sum_{j=1}^r q_j = 1$, so that $|\vec{n}|/n \rightarrow 1$ as $n \rightarrow \infty$. Consider Poisson distributions with parameters $(a_1, a_2, \dots, a_{r-1}, Na_r)$, i.e., the last parameter grows linearly with N . Then for $t > 0$ one has*

$$\lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{C_{\vec{n}+\vec{e}_r}(Nx)}{NC_{\vec{n}}(Nx)} = \frac{x - a_r - t + \sqrt{(x - a_r - t)^2 - 4a_r q_r t}}{2} := g_r(x) \quad (5.1)$$

and for $1 \leq k < r$

$$\lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{C_{\vec{n}+\vec{e}_k}(Nx)}{NC_{\vec{n}}(Nx)} = x - t - \frac{a_r q_r t}{g_r(x)} \quad (5.2)$$

uniformly on compact sets of $\mathbb{C} \setminus [0, \infty)$.

Proof. We still use the notation $P_{\vec{n},N}(x) = C_{\vec{n}}(Nx)/N^{|\vec{n}|}$, but now keep in mind that $C_{\vec{n}}$ depends on the r parameters $(a_1, \dots, a_{r-1}, Na_r)$ so that the parameter N appears not only in the scaling of the variable (Nx) but also in the last parameter (Na_r). The recurrence relation (2.4), after dividing by $C_{\vec{n}}(Nx)$ gives for $x \in K$, where K is a compact set in $\mathbb{C} \setminus [0, \infty)$,

$$x = \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} + \frac{a_k + |\vec{n}|}{N} + \sum_{j=1}^{r-1} \frac{n_j a_j}{N^2} \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} + \frac{n_r a_r}{N} \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)}$$

when $1 \leq k \leq r-1$, and for $k = r$ we have

$$x = \frac{P_{\vec{n}+\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} + \frac{Na_r + |\vec{n}|}{N} + \sum_{j=1}^{r-1} \frac{n_j a_j}{N^2} \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} + \frac{n_r a_r}{N} \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)}.$$

If we use (3.2), then for $1 \leq k \leq r-1$

$$\left| \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} - x + \frac{a_k + |\vec{n}|}{N} + \frac{n_r a_r}{N} \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} \right| \leq \frac{1}{\delta N^2} \sum_{j=1}^{r-1} n_j a_j$$

and for $k = r$

$$\left| \frac{P_{\vec{n}+\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} - x + \frac{Na_r + |\vec{n}|}{N} + \frac{n_r a_r}{N} \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} \right| \leq \frac{1}{\delta N^2} \sum_{j=1}^{r-1} n_j a_j$$

so that

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \left| \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} - x + t + a_r q_r t \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} \right| = 0, \quad 1 \leq k \leq r-1,$$

and

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \left| \frac{P_{\vec{n}+\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} - x + a_r + t + a_r q_r t \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} \right| = 0,$$

uniformly on K . The bound (3.2) implies that $\{P_{\vec{n}-\vec{e}_j,N}(x)/P_{\vec{n},N}(x) : n, N \in \mathbb{N}\}$ is a normal family on every compact subset of $\mathbb{C} \setminus [0, \infty)$, hence there is a subsequence which converges uniformly on K :

$$\lim_{n_i \rightarrow \infty, n_i/N_i \rightarrow t} \frac{P_{\vec{n}_i-\vec{e}_j,N_i}(x)}{P_{\vec{n}_i,N_i}(x)} = h_j(x),$$

and, by taking further subsequences, this convergence holds for every j for which $1 \leq j \leq r$. With our previous estimates, this gives

$$\lim_{n_i \rightarrow \infty, n_i/N_i \rightarrow t} \frac{P_{\vec{n}_i+\vec{e}_j,N_i}(x)}{P_{\vec{n}_i,N_i}(x)} = x - t - a_r q_r t h_r(x), \quad 1 \leq j \leq r-1, \quad (5.3)$$

and

$$\lim_{n_i \rightarrow \infty, n_i/N_i \rightarrow t} \frac{P_{\vec{n}_i+\vec{e}_r,N_i}(x)}{P_{\vec{n}_i,N_i}(x)} = x - a_r - t - a_r q_r t h_r(x). \quad (5.4)$$

A technical estimation (see Lemma 5.1 at the end of this section) implies that

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \left| \frac{P_{\vec{n},N}(x)}{P_{\vec{n}+\vec{e}_j,N}(x)} - \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} \right| = 0, \quad 1 \leq j \leq r,$$

uniformly on K , hence (5.4) gives

$$\frac{1}{h_r(x)} = x - a_r - t - a_r q_r t h_r(x).$$

If we put $g_r(x) = 1/h_r(x)$, then this gives a quadratic equation for $g_r(x)$, with solutions

$$\frac{x - a_r - t \pm \sqrt{(x - a_r - t)^2 - 4a_r q_r t}}{2}.$$

Since $h_r(x) = 1/x + \mathcal{O}(1/x^2)$, we need to choose the solution with the positive sign for $g_r(x)$. This limit is independent of the subsequence that we selected, hence every convergent subsequence has the same limit, which implies that the full sequence converges to this limit. This gives (5.1), and by using (5.3) we easily find (5.2). \square

The limit function $g_r(x)$ is the solution of a quadratic equation. In general, if $k \leq r$ of the parameters grow linearly with N , then the limit function is expected to be the solution, which grows as x when $x \rightarrow \infty$, of an algebraic equation of degree $k+1$.

For the asymptotic behavior of the zeros we have

Theorem 5.2. *Suppose $n_j = \lfloor q_j n \rfloor$, with $0 < q_j < 1$ and $\sum_{j=1}^r q_j = 1$, so that $|\vec{n}|/n \rightarrow 1$ if $n \rightarrow \infty$. Consider Poisson distributions with parameters $(a_1, a_2, \dots, a_{r-1}, Na_r)$, i.e., the last parameter grows linearly with N . Then for $t > 0$ one has*

$$\lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{1}{|\vec{n}|} \sum_{j=1}^{|\vec{n}|} f(x_{\vec{n},j}/N) = \frac{1}{t} \int_0^{(1-q_r)t} f(x) dx + q_r \int_{\alpha_t}^{\beta_t} v(x) f(x) dx$$

for every bounded continuous function f on $[0, \infty)$, where $(1 - q_r)t \leq \alpha_t < \beta_t$ and v is a probability density on $[\alpha_t, \beta_t]$.

Proof. We can start from equation (4.3):

$$\frac{P'_{\vec{n},N}(x)}{P_{\vec{n},N}(x)} = \frac{P'_{\vec{n}-n_r\vec{e}_r,N}(x)}{P_{\vec{n}-n_r\vec{e}_r,N}(x)} + \sum_{k=0}^{n_r-1} \left(\frac{P_{\vec{n}-k\vec{e}_r,N}(x)}{P_{\vec{n}-(k+1)\vec{e}_r,N}(x)} \right)' / \left(\frac{P_{\vec{n}-k\vec{e}_r,N}(x)}{P_{\vec{n}-(k+1)\vec{e}_r,N}(x)} \right).$$

The multiple orthogonal polynomial $P_{\vec{n}-n_r\vec{e}_r}$ is in fact the multiple Charlier polynomial with the $r-1$ parameters (a_1, \dots, a_{r-1}) , which do not depend on N . Hence we can use Theorem 4.1 which gives (4.4). We write the sum as an integral, as we did in the proof of Theorem 4.1, but now we use Theorem 5.1 to find

$$\lim_{n,N \rightarrow \infty, n/N \rightarrow t} \frac{1}{n_r} \sum_{k=0}^{n_r-1} \left(\frac{P_{\vec{n}-k\vec{e}_r,N}(x)}{P_{\vec{n}-(k+1)\vec{e}_r,N}(x)} \right)' / \left(\frac{P_{\vec{n}-k\vec{e}_r,N}(x)}{P_{\vec{n}-(k+1)\vec{e}_r,N}(x)} \right) = \int_0^1 \frac{g_r(x, s)'}{g_r(x, s)} ds$$

where

$$g_r(x, s) = \frac{x - a_r - (1 - q_r s)t + \sqrt{(x - a_r - (1 - q_r s)t)^2 - 4a_r q_r(1 - s)t}}{2}$$

and the prime is the derivative d/dx . The $g_r(x, s)$ is obtained from Theorem 5.1 after the substitutions

$$q_j \rightarrow \frac{q_j}{1 - q_r s}, \quad 1 \leq j \leq r-1, \quad q_r \rightarrow \frac{(1-s)q_r}{1 - q_r s},$$

so that $\sum_{j=1}^r q_j = 1$,

$$n_r \rightarrow n_r - \lfloor n_r s \rfloor, \quad n \rightarrow n(1 - q_r s), \quad t \rightarrow (1 - q_r s)n.$$

Observe that

$$\frac{g_r(x, s)'}{g_r(x, s)} = \frac{1}{\sqrt{(x - a_r - (1 - q_r s)t)^2 - 4a_r q_r(1 - s)t}}$$

and if we use the well known Stieltjes transform

$$\frac{1}{\sqrt{x^2 - 1}} = \frac{1}{\pi} \int_{-1}^1 \frac{1}{x - y} \frac{dy}{\sqrt{1 - y^2}}, \quad x \in \mathbb{C} \setminus [-1, 1],$$

then one finds

$$\frac{g_r(x, s)'}{g_r(x, s)} = \frac{1}{\pi} \int_{\alpha(s)}^{\beta(s)} \frac{1}{x - y} \frac{dy}{\sqrt{4a_r q_r(1 - s)t - (y - a_r - (1 - q_r s)t)^2}},$$

where

$$\alpha(s) = a_r + (1 - q_r s)t - 2\sqrt{a_r q_r(1 - s)t}, \quad \beta(s) = a_r + (1 - q_r s)t + 2\sqrt{a_r q_r(1 - s)t}.$$

In order to write

$$\int_0^1 \frac{g_r(x, s)'}{g_r(x, s)} ds$$

as a Stieltjes transform, we need to change the order of integration in

$$\int_0^1 \frac{1}{\pi} \int_{\alpha(s)}^{\beta(s)} \frac{1}{x - y} \frac{dy}{\sqrt{4a_r q_r(1 - s)t - (y - a_r - (1 - q_r s)t)^2}} ds. \quad (5.5)$$

Observe that

$$\alpha(0) = a_r + t - 2\sqrt{a_r q_r t}, \quad \beta(0) = a_r + t + 2\sqrt{a_r q_r t}, \quad \alpha(1) = \beta(1) = a_r + (1 - q_r)t,$$

and that the function β is monotonically decreasing for $s \in [0, 1]$. We need to distinguish between two cases.

Case 1: $a_r \geq q_r t$. In this case the function α is monotonically increasing for $s \in [0, 1]$, see Figure 1.

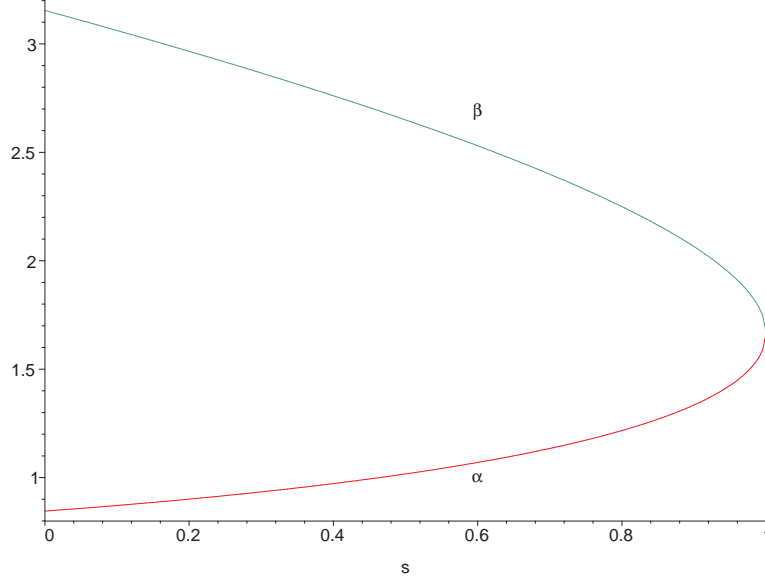


Figure 1: The functions α and β for case 1

If we define $\alpha_t = \alpha(0)$ and $\beta_t = \beta(0)$ then

$$\begin{cases} \alpha^{-1}(y) = \frac{-y - a_r + t + 2\sqrt{a_r(y - (1 - q_r)t)}}{q_r t}, & \alpha_t \leq y \leq a_r + (1 - q_r)t \\ \beta^{-1}(y) = \frac{-y - a_r + t + 2\sqrt{a_r(y - (1 - q_r)t)}}{q_r t}, & a_r + (1 - q_r)t \leq y \leq \beta_t \end{cases}$$

so that interchanging the order of integration in (5.5) gives

$$\frac{1}{\pi} \int_{\alpha_t}^{\beta_t} \frac{dy}{x - y} \int_0^{\frac{-y - a_r + t + 2\sqrt{a_r(y - (1 - q_r)t)}}{q_r t}} \frac{ds}{\sqrt{4a_r q_r (1 - s)t - (y - a_r - (1 - q_r s)t)^2}}.$$

When we change the variable s to a new variable u by

$$s = \frac{-y - a_r + t + 2u\sqrt{a_r(y - (1 - q_r)t)}}{q_r t},$$

then the integral simplifies to

$$\frac{1}{\pi q_r t} \int_{\alpha_t}^{\beta_t} \frac{dy}{x - y} \int_{\frac{y + a_r - t}{2\sqrt{a_r(y - (1 - q_r)t)}}}^1 \frac{du}{\sqrt{1 - u^2}}.$$

This gives the weight function

$$v(y) = \frac{1}{\pi q_r t} \int_{\frac{y+a_r-t}{2\sqrt{a_r(y-(1-q_r)t)}}}^1 \frac{du}{\sqrt{1-u^2}}, \quad \alpha_t \leq y \leq \beta_t.$$

An easy exercise gives that $(1-q_r)t \leq \alpha_t < \beta_t$.

Case 2: $a_r < q_r t$. In this case α has a global minimum on $]0, 1[$ at $s = 1 - a_r/q_r t$, and the minimum is $(1-q_r)t$, see Figure 2.

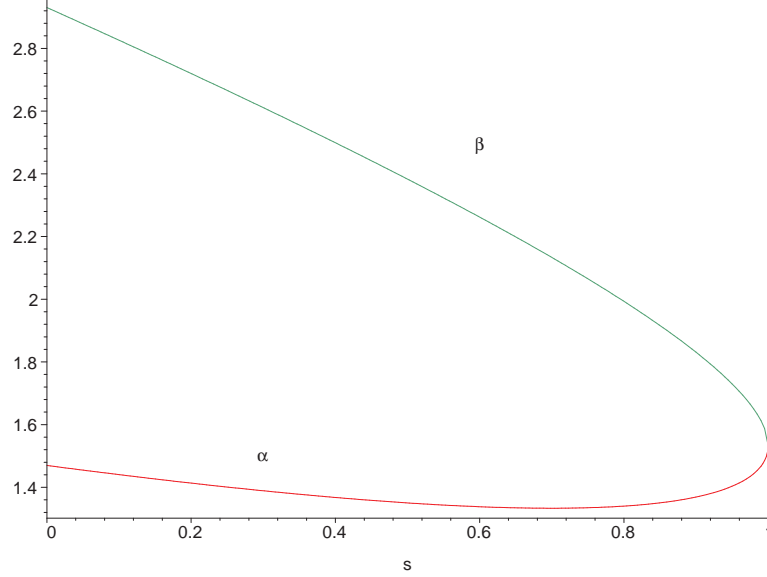


Figure 2: The functions α and β for case 2

Interchanging the order of the integrals in (5.5) now gives two pieces

$$\begin{aligned} & \frac{1}{\pi} \int_{(1-q_r)t}^{\alpha(0)} \frac{dy}{x-y} \int_{\frac{-y-a_r+t-2\sqrt{a_r(y-(1-q_r)t)}}{q_r t}}^{\frac{-y-a_r+t+2\sqrt{a_r(y-(1-q_r)t)}}{q_r t}} \frac{ds}{\sqrt{4a_r q_r (1-s)t - (y-a_r - (1-q_r s)t)^2}} \\ & + \frac{1}{\pi} \int_{\alpha(0)}^{\beta(0)} \frac{dy}{x-y} \int_0^{\frac{-y-a_r+t+2\sqrt{a_r(y-(1-q_r)t)}}{q_r t}} \frac{ds}{\sqrt{4a_r q_r (1-s)t - (y-a_r - (1-q_r s)t)^2}}. \end{aligned}$$

The change of variable $s \rightarrow u$ with

$$s = \frac{-y-a_r+t+2u\sqrt{a_r(y-(1-q_r)t)}}{q_r t}$$

now gives

$$\frac{1}{\pi q_r t} \int_{(1-q_r)t}^{\alpha(0)} \frac{dy}{x-y} \int_{-1}^1 \frac{du}{\sqrt{1-u^2}} + \frac{1}{\pi q_r t} \int_{\alpha(0)}^{\beta(0)} \frac{dy}{x-y} \int_{\frac{y+a_r-t}{2\sqrt{a_r(y-(1-q_r)t)}}}^1 \frac{du}{\sqrt{1-u^2}}.$$

So if we now define $\alpha_t = (1-q_r)t$ and $\beta_t = \beta(0)$, then obviously $(1-q_r)t = \alpha_t < \beta_t$ and the weight function becomes

$$v(y) = \frac{1}{q_r t}, \quad (1-q_r)t \leq y \leq \alpha(0)$$

and

$$v(y) = \frac{1}{\pi q_r t} \int_{\frac{y+a_r-t}{2\sqrt{a_r(y-(1-q_r)t)}}}^1 \frac{du}{\sqrt{1-u^2}}, \quad \alpha(0) \leq y \leq \beta(0).$$

So in both cases we get

$$\lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{1}{n_r} \sum_{k=0}^{n_r-1} \left(\frac{P_{\vec{n}-k\vec{e}_r, N}(x)}{P_{\vec{n}-(k+1)\vec{e}_r, N}(x)} \right)' / \left(\frac{P_{\vec{n}-k\vec{e}_r, N}(x)}{P_{\vec{n}-(k+1)\vec{e}_r, N}(x)} \right) = \int_{\alpha_t}^{\beta_t} \frac{v(y)}{x-y} dy,$$

and combining this with (4.4) gives

$$\lim_{n, N \rightarrow \infty, n/N \rightarrow t} \frac{1}{|\vec{n}|} \frac{P'_{\vec{n}, N}(x)}{P_{\vec{n}, N}(x)} = \frac{1}{t} \int_0^{(1-q_r)t} \frac{1}{x-y} dy + q_r \int_{\alpha_t}^{\beta_t} \frac{v(y)}{x-y} dy,$$

which gives the desired result in view of the Grommer-Hamburger theorem [6]. \square

The first portion of $(1-q_r)n$ of the zeros of $C_{\vec{n}}(Nx)$ are uniformly distributed on $[0, (1-q_r)t]$ and hence the constraint that ‘between two positive integers there can be at most one zero’ is in action and the zeros are forced to approach the first $(1-q_r)n$ integers in \mathbb{N} . If $a_r \geq q_r t$ (case 1) then the last portion of $q_r n$ of the zeros have a different distribution on an interval $[\alpha_t, \beta_t] = [a_r + t - 2\sqrt{a_r q_r t}, a_r + t + 2\sqrt{a_r q_r t}]$ to the right of the interval $[0, (1-q_r)t]$ where the other zeros accumulate. This means that those last $q_r n$ zeros are less dense distributed and some of the intervals between two integers may be free of zeros. If $a_r < q_r t$ (case 2) then some of the $q_r n$ last zeros are still uniformly distributed on $[(1-q_r)t, a_r + t - 2\sqrt{a_r q_r t}]$ but the remaining zeros are less dense distributed on $[a_r + t - 2\sqrt{a_r q_r t}, a_r + t + 2\sqrt{a_r q_r t}]$ and this interval now touches the interval where the zeros are uniformly distributed. In fact, a transition occurs when $a_r = q_r t$ in the sense that the (scaled) zeros have a zero distribution on two disjoint intervals when $a_r > q_r t$ and the zero distribution is supported on one interval when $a_r < q_r t$. Moreover, since

$$\int_z^1 \frac{du}{\sqrt{1-u^2}} \sim C\sqrt{1-z}, \quad z \rightarrow 1-$$

and for $a_r > q_r t$

$$1 - \frac{y + a_r - t}{2\sqrt{a_r(y - (1-q_r)t)}} \sim \begin{cases} C_1(y - \alpha(0)), & y \rightarrow \alpha(0)+ \\ C_2(\beta(0) - y), & y \rightarrow \beta(0)- \end{cases}$$

we see that the density v near the endpoints $\alpha(0)$ and $\beta(0)$ tends to zero as $\sqrt{y - \alpha(0)}$ and $\sqrt{\beta(0) - y}$, respectively (see Figure 3, picture on the left, for $a_r = 1$, $q_r = 1/10$ and $t = 1$).

For $a_r < q_r t$ we still have $v(y) \sim C\sqrt{\beta(0) - y}$ near the endpoint $\beta(0)$. The transition from uniform density to non-uniform density occurs at $y = \alpha(0)$, but now

$$\frac{y + a_r - t}{2\sqrt{a_r(y - (1-q_r)t)}} \rightarrow -1, \quad y \rightarrow \alpha(0)+$$

so that $v(y) \rightarrow 1/q_r t$ as $y \rightarrow \alpha(0)+$, and the density is continuous at the transition point $\alpha(0)$ (see Figure 3, picture on the right, for $a_r = 1/10$, $q_r = 1/5$ and $t = 1$).

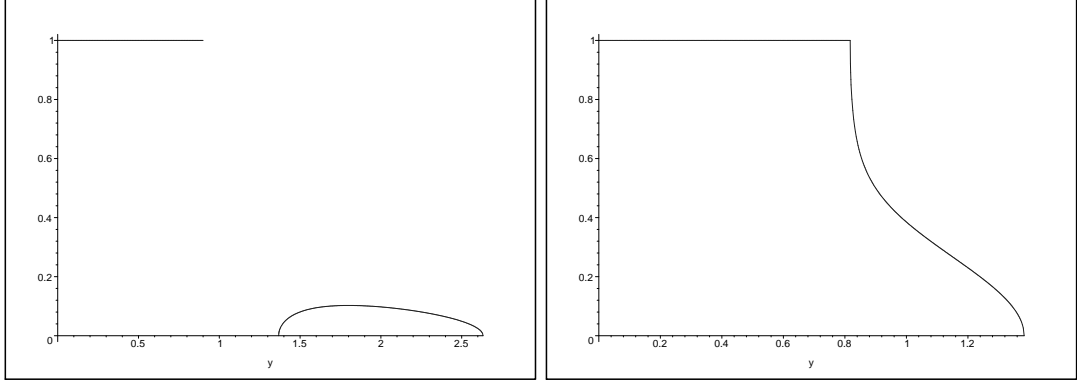


Figure 3: The density of the (scaled) zeros of multiple Charlier polynomials

When $a_r = q_r t$ we have

$$\frac{y + a_r - t}{2\sqrt{a_r(y - (1 - q_r)t)}} \rightarrow 0, \quad y \rightarrow \alpha(0) +$$

so that $v(y) \rightarrow 1/2q_r t$ as $y \rightarrow \alpha(0) +$, so that the density is not continuous at the transition point.

Such transitions also occur when $k < r$ of the parameters depend linearly on N . In that case the zeros of $C_{\vec{n}}(Nx)$ may accumulate on at most $k + 1$ disjoint intervals. If all the parameters depend on N (i.e., $k = r$) then the zeros accumulate on at most r disjoint intervals. The analysis for $k > 1$ is more involved since this involves algebraic functions of order $k + 1$.

One technical, but crucial, step in the proof of Theorem 5.1 is the following.

Lemma 5.1. *Let $P_{\vec{n},N}(x) = C_{\vec{n}}(Nx)/N^{|\vec{n}|}$, where $C_{\vec{n}}$ are the multiple Charlier polynomials with parameters $(a_1, \dots, a_{r-1}, Na_r)$. Let K be a compact set in $\mathbb{C} \setminus [0, \infty)$, then for every k and ℓ with $1 \leq k, \ell \leq r$ one has, uniformly for $x \in K$*

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} \left| \frac{P_{\vec{n},N}(x)}{P_{\vec{n}+\vec{e}_k,N}(x)} - \frac{P_{\vec{n}-\vec{e}_\ell,N}(x)}{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)} \right| = 0.$$

Proof. From the recurrence relation, we have

$$x = \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} + \frac{a_k + |\vec{n}|}{N} + \sum_{j=1}^{r-1} \frac{n_j a_j}{N^2} \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} + \frac{n_r a_r}{N} \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)}$$

when $1 \leq k \leq r - 1$, and for $k = r$ we have

$$x = \frac{P_{\vec{n}+\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} + \frac{Na_r + |\vec{n}|}{N} + \sum_{j=1}^{r-1} \frac{n_j a_j}{N^2} \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)} + \frac{n_r a_r}{N} \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)}.$$

We will denote

$$E_{\vec{n}}(x) = \sum_{j=1}^{r-1} \frac{n_j a_j}{N^2} \frac{P_{\vec{n}-\vec{e}_j,N}(x)}{P_{\vec{n},N}(x)}$$

and the bound (3.2) then gives

$$|E_{\vec{n}}(x)| \leq \frac{1}{\delta N^2} \sum_{j=1}^{r-1} n_j a_j \leq \frac{C|\vec{n}|}{\delta N^2},$$

where $C > 0$ is a constant (in fact one may take $\max_{1 \leq j \leq r-1} a_j$). If we change \vec{n} to $\vec{n} - \vec{e}_\ell$, then

$$x = \frac{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} + \frac{a_k + |\vec{n}| - 1}{N} + E_{\vec{n}-\vec{e}_\ell}(x) + \frac{(n_r - \delta_{r,\ell})a_r}{N} \frac{P_{\vec{n}-\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)}$$

when $1 \leq k \leq r-1$, and for $k = r$

$$x = \frac{P_{\vec{n}+\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} + \frac{Na_r + |\vec{n}| - 1}{N} + E_{\vec{n}-\vec{e}_\ell}(x) + \frac{(n_r - \delta_{r,\ell})a_r}{N} \frac{P_{\vec{n}-\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)}.$$

If we subtract the equations for $\vec{n} - \vec{e}_\ell$ from those with \vec{n} , then we find

$$\begin{aligned} 0 = & \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} - \frac{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} + \frac{1}{N} + E_{\vec{n}}(x) - E_{\vec{n}-\vec{e}_\ell}(x) \\ & + \frac{n_r a_r}{N} \left(\frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} - \frac{P_{\vec{n}-\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} \right) + \frac{\delta_{r,\ell} a_r}{N} \frac{P_{\vec{n}-\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)}. \end{aligned}$$

We have

$$|E_{\vec{n}}(x) - E_{\vec{n}-\vec{e}_\ell}(x)| \leq \frac{2C|\vec{n}|}{\delta N^2}$$

and we will take $|\vec{n}| \leq C_2 N$, therefore we have

$$\left| \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} - \frac{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} \right| \leq \frac{C_1}{N\delta} + C_2 a_r \left| \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} - \frac{P_{\vec{n}-\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} \right|,$$

where C_1 and C_2 are constants. If we use the bound (3.2), then

$$\left| \frac{P_{\vec{n}+\vec{e}_k,N}(x)}{P_{\vec{n},N}(x)} - \frac{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} \right| \geq \delta^2 \left| \frac{P_{\vec{n},N}(x)}{P_{\vec{n}+\vec{e}_k,N}(x)} - \frac{P_{\vec{n}-\vec{e}_\ell,N}(x)}{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)} \right|$$

so that

$$\left| \frac{P_{\vec{n},N}(x)}{P_{\vec{n}+\vec{e}_k,N}(x)} - \frac{P_{\vec{n}-\vec{e}_\ell,N}(x)}{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)} \right| \leq \frac{C_1}{N\delta^3} + \frac{C_2 a_r}{\delta^2} \left| \frac{P_{\vec{n}-\vec{e}_r,N}(x)}{P_{\vec{n},N}(x)} - \frac{P_{\vec{n}-\vec{e}_r-\vec{e}_\ell,N}(x)}{P_{\vec{n}-\vec{e}_\ell,N}(x)} \right|.$$

If we use the notation

$$D_{\vec{n},k,\ell} = \left| \frac{P_{\vec{n},N}(x)}{P_{\vec{n}+\vec{e}_k,N}(x)} - \frac{P_{\vec{n}-\vec{e}_\ell,N}(x)}{P_{\vec{n}+\vec{e}_k-\vec{e}_\ell,N}(x)} \right|,$$

then this gives

$$D_{\vec{n},k,\ell} \leq \frac{C_1}{N\delta^3} + \frac{C_2 a_r}{\delta^2} D_{\vec{n}-\vec{e}_r,r,\ell}.$$

Put

$$D_{\vec{n},\ell} = \max_{1 \leq k \leq r} D_{\vec{n},k,\ell},$$

then one has

$$D_{\vec{n},\ell} \leq \frac{C_1}{N\delta^3} + \frac{C_2 a_r}{\delta^2} D_{\vec{n}-\vec{e}_r,\ell}.$$

Iterating this inequality gives

$$D_{\vec{n},\ell} \leq \left(\frac{C_2 a_r}{\delta^2} \right)^{n_r} D_{\vec{n}-n_r \vec{e}_r,\ell} + \frac{C_1}{N\delta^3} \sum_{j=0}^{n_r-1} \left(\frac{C_2 a_r}{\delta^2} \right)^j.$$

Now choose a compact K' (with an accumulation point) far enough from $[0, \infty)$ so that δ is large and $C_2 a_r / \delta^2 < 1$. Then for $x \in K'$

$$D_{\vec{n},\ell} \leq \left(\frac{C_2 a_r}{\delta^2} \right)^{n_r} D_{\vec{n}-n_r \vec{e}_r,\ell} + \frac{C_1}{N\delta^3} \frac{1}{1 - C_2 a_r / \delta^2}.$$

The bound (3.2) gives $D_{\vec{n},\ell} \leq 2/\delta$, hence if we put $\vec{n} = (\lfloor nq_1 \rfloor, \dots, \lfloor nq_r \rfloor)$ and let $n, N \rightarrow \infty$ such that $n/N \rightarrow t > 0$, then

$$\lim_{n \rightarrow \infty, n/N \rightarrow t} D_{\vec{n},\ell} = 0$$

uniformly for $x \in K'$. So we have convergence of $D_{\vec{n},\ell} \rightarrow 0$ uniformly on a set K' with an accumulation point, but then Vitali's theorem implies that $D_{\vec{n},\ell}$ converges to zero uniformly on every compact K where a bound (3.2) holds, hence for $K \subset \mathbb{C} \setminus [0, \infty)$. \square

6 Concluding remarks

In this paper we have investigated the ratio asymptotic behavior of the multiple Charlier polynomials and from it we obtained the asymptotic distribution of the zeros, after proper rescaling. The next step is to find the asymptotic behavior of the polynomials $C_{\vec{n}}$ themselves: the strong asymptotic behavior or the uniform asymptotic behavior. As in the case of the usual Charlier polynomials, one will need to look at different regions in the complex plane: away from the positive real line, on the oscillatory region where all the zeros are, near the largest zero, near the origin, etc. One way to do this is to use the integral relation which can be obtained from the multivariate generating function and to apply a steepest descent analysis (but for a multiple integral), as was done by Goh [7] and Rui and Wong [20] for Charlier polynomials. Another way is to use the Riemann-Hilbert problem (for $(r+1) \times (r+1)$ matrices) and the steepest descent method for oscillatory Riemann-Hilbert problems, as was done by Ou and Wong [17] for Charlier polynomials. One of the steps in that asymptotic analysis is to transform the Riemann-Hilbert problem to a normalized (at infinity) Riemann-Hilbert problem, and this requires g -functions which are logarithmic potentials of the asymptotic zero distribution. Hence the results in Section 4 and Section 5 (in particular Theorem 5.2) will be needed.

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